*J. Fluid Mech.* (2007), *vol.* 580, *pp.* 431–450.  
doi:10.1017/S0022112007005733 Printed in the United Kingdom

# On the large-scale structure of homogeneous two-dimensional turbulence

P. A. DAVIDSON
Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK

(Received 28 September 2006 and in revised form 12 January 2007)

We consider freely decaying two-dimensional isotropic turbulence. It is usually assumed that, in such turbulence, the energy spectrum at small wavenumber, $k$, takes the form $E(k \to 0) \sim Ik^3$, where $I$ is the two-dimensional version of Loitsyansky's integral. However, a second possibility is $E(k \to 0) \sim Lk$, where the pre-factor, $L$, is the two-dimensional analogue of Saffman's integral. We show that, as in three dimensions, $L$ is an invariant and that $E \sim Lk$ spectra arise whenever the eddies possess a significant amount of linear impulse. The conservation of $L$ is shown to be a direct consequence of the principle of conservation of linear momentum. We also show that isotropic turbulence dominated by a cloud of randomly located monopole vortices has a singular energy spectrum of the form $E(k \to 0) \sim Jk^{-1}$, where $J$, like $L$, is an invariant. However, while $E \sim Jk^{-1}$ necessarily implies the existence of a sea of monopoles, the converse need not be true: a sea of monopoles whose spatial locations are not entirely random, but constrained in some way, need not give a $E \sim Jk^{-1}$ spectra. The constraint imposed by the conservation of energy is particularly important, ruling out $E \sim Jk^{-1}$ spectra for certain classes of initial conditions. Finally, we provide simple explicit examples of random vorticity fields with $E \sim Ik^3$, $E \sim Lk$ and $E \sim Jk^{-1}$ spectra.

## 1. Introduction

### 1.1. *The large-scale structure of three-dimensional turbulence*

In order to place this work in context, we start by considering three-dimensional, rather than two-dimensional, turbulence. In three-dimensional isotropic turbulence, the form of the energy spectrum at small wavenumber, $k$, takes the form,

$$E(k) = Lk^2/4\pi^2 + Ik^4/24\pi^2 + \cdots, \tag{1.1}$$

provided that the two-point velocity correlation, $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$, decays sufficiently rapidly with separation $r = |\boldsymbol{r}| = |\boldsymbol{x}' - \boldsymbol{x}|$. (Here $\boldsymbol{u}$ is measured at $\boldsymbol{x}$ and $\boldsymbol{u}'$ at $\boldsymbol{x}'$.) The pre-factors, $L$ and $I$, are known as the Saffman and Loitsyansky integrals, respectively, and can be written as,

$$L = \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r}, \tag{1.2}$$

and

$$I = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r}. \tag{1.3}$$

(See, for example, Davidson 2004, § 6.3.) Now $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ is related to the longitudinal correlation function, $f(r)$, by

$$\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle = \frac{u^2}{r^2} \frac{\partial}{\partial r}(r^3 f), \quad u^2 = \tfrac{1}{3}\langle \boldsymbol{u}^2 \rangle, \tag{1.4}$$

and so Saffman's integral can be rewritten as

$$L = 4\pi u^2 [r^3 f]_\infty. \tag{1.5}$$

(The subscript $\infty$ indicates $r \to \infty$.) Evidently, whether or not we obtain a Saffman, $E \sim Lk^2$, spectrum depends on the asymptotic form of $f$ at large $r$. Alternatively, noting that ensemble and volume averages are equivalent, we may rewrite $L$ in the form,

$$L = \left\langle \left[ \int_V \boldsymbol{u} \, \mathrm{d}V \right]^2 \right\rangle \Big/ V, \tag{1.6}$$

where $V$ is some large volume. Thus we obtain a Saffman spectrum when the turbulence contains a sufficiently large amount of linear momentum (Saffman 1967), and this occurs when the eddies (blobs of vorticity, $\boldsymbol{\omega}$) typically have a finite amount of linear impulse, $\frac{1}{2}\int \boldsymbol{x} \times \boldsymbol{\omega} \, \mathrm{d}V$ (see, for example, Davidson 2004, § 6.3). Saffman also showed that $L$ is an invariant of freely decaying three-dimensional turbulence.

When $L$ is zero, which occurs if the turbulence has insufficient linear impulse, we obtain a Batchelor spectrum, $E \sim Ik^4$. Landau showed that, provided the two-point velocity correlations decay sufficiently rapidly with distance (and they probably do not in practice), then $I$ can be related to the angular momentum of the turbulence as follows:

$$I = \operatorname*{Lim}_{V \to \infty} \frac{\langle \boldsymbol{H}^2 \rangle}{V}, \quad \boldsymbol{H} = \int_V (\boldsymbol{x} \times \boldsymbol{u}) \, \mathrm{d}V. \tag{1.7}$$

Thus there are some similarities between $I$ and $L$. However, unlike $L$, $I$ is not, in general, an invariant. That is, the Kármán–Howarth equation integrates to give

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 8\pi [u^3 r^4 K]_\infty, \tag{1.8}$$

where $K$ is the usual triple longitudinal correlation function, $u^3 K = \langle u_x^2(\boldsymbol{x}) u_x(\boldsymbol{x} + r\hat{\boldsymbol{e}}_x) \rangle$, and the work of Batchelor & Proudman (1956) suggests that the long-range pressure forces will, in general, establish long-range correlations of the from $K_\infty \sim r^{-4}$. Certainly, numerical simulations of $E \sim Ik^4$ turbulence usually show a slow rise in $I$. Curiously, though, recent simulations performed in very large computational domains show $I \approx$ constant once the turbulence has become fully developed (Ishida, Davidson & Kaneda 2006).

Both $E \sim Lk^2$ and $E \sim Ik^4$ types of turbulence are readily generated in computer simulations. Which form is seen depends on the initial conditions. If $L$ is non-zero at $t = 0$, then we obtain a Saffman spectrum, whereas $L = 0$ at $t = 0$ excludes such a spectrum. Opinion is divided, however, as to whether grid turbulence, for example, is of the Saffman or Batchelor type, with Saffman (1967) suggesting it is of the form $E \sim Ik^4$.

Note that $E \sim c_n k^n$, $2 < n < 4$, is also a theoretical possibility for homogeneous turbulence, with $c_n =$ constant for $n < 4$. However, since the spectral tensor is singular for $n \neq 4$, it is usually thought that Saffman and Batchelor spectra are the most natural cases.

We close this section with a comment about the relationship between homogeneous isotropic turbulence and numerical simulations in a periodic cube. The question of interest is whether or not periodicity excludes certain statistical states, such as Saffman turbulence. We raise this issue here because it is relevant to our subsequent discussion of two-dimension turbulence, yet the numerical evidence is more extensive in three dimensions.

Now it is clear that, if the domain size, $L_{BOX}$, in a periodic simulation is not much larger than the integral scale, $\ell$, of the turbulence, then the large scales can behave quite differently to those of conventional homogeneous turbulence. After all, periodicity enforces an unphysical boundary condition, as well as anisotropy, on the large scales (see, for example, Davidson 2004, § 7.2). Nevertheless, it seems reasonable to suppose that flow in a periodic cube approaches that of conventional homogeneous turbulence as $L_{BOX}/\ell \to \infty$. It follows that, since a Saffman spectrum is a legitimate state for homogeneous turbulence, then it should also be realizable in a large periodic cube, and indeed there are many examples of this in the literature (see, for example, Lesieur, Ossai & Metais 1999).

At first sight this appears paradoxical, since $\int \boldsymbol{u}\,\mathrm{d}V = 0$ in a periodic cube, provided the integral is taken over the whole domain, and this seems to be at odds with (1.6). There is no such contradiction, however, as can be seen from the following argument. Consider first strictly homogenous isotropic turbulence. Here we enforce $\langle \boldsymbol{u} \rangle = 0$ through a suitable choice of frame of reference, and this requires

$$\operatorname*{Lim}_{V\to\infty} \frac{\int \boldsymbol{u}\,\mathrm{d}V}{V} = 0. \tag{1.9}$$

However, this is not enough to enforce the stronger condition

$$L = \operatorname*{Lim}_{V\to\infty} \frac{\left(\int \boldsymbol{u}\,\mathrm{d}V\right)^2}{V} = 0, \tag{1.10}$$

and indeed the central limit theorem suggests that $\int \boldsymbol{u}\,\mathrm{d}V \sim V^{1/2}$, so that we might expect that $L$ is, in general, non-zero. (That is, we may consider the turbulence to be composed of a sea of eddies, each of which has some linear impulse. Now the linear momentum within some large spherical volume $V$ is proportional to the sum of the linear impulses of the individual eddies contained within $V$, and if the eddies are assigned a random linear impulse taken from a p.d.f. of zero mean, then (1.9) is satisfied, but (1.10) is not.) Thus, a Saffman spectrum is quite natural, provided that the eddies possess a significant amount of linear impulse.

Turning now to a period cube in which $L_{BOX}/\ell \gg 1$, the fact that $\int \boldsymbol{u}\,\mathrm{d}V = 0$ for the domain as a whole is equivalent to condition (1.9) in homogeneous turbulence. However, if we consider a large volume of radius $R$, such that $\ell \ll R \ll L_{BOX}$, then in general $\int \boldsymbol{u}\,\mathrm{d}V$ will be non-zero for that volume, and indeed we would expect $\int \boldsymbol{u}\,\mathrm{d}V \sim V^{1/2} \sim R^{3/2}$, provided the eddies have a significant linear impulse. Thus, periodicity does not exclude a Saffman spectrum.

We shall see that similar issues arise in our discussion of two-dimensional turbulence in a periodic square, where the fact that $\int \boldsymbol{u}\,\mathrm{d}V = 0$, or $\int \omega\,\mathrm{d}V = 0$, $\omega$ being the vorticity, does not exclude the possibility that $\int \langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle\,\mathrm{d}\boldsymbol{r}$, or $\int \langle \omega\omega' \rangle\,\mathrm{d}\boldsymbol{r}$, are finite.

### 1.2. *From three dimensions to two dimensions*

Let us now turn to two-dimensional isotropic turbulence. We shall see, in § 2, that the analogue of (1.1) is

$$E(k) = Lk/4\pi + Ik^3/16\pi + \cdots, \tag{1.11}$$

where, as before,

$$L = \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r}, \quad I = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r}, \tag{1.12}$$

and we require $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ to decay sufficiently rapidly with $|\boldsymbol{r}|$ for the integrals to exist. (Of course, the integration over $\boldsymbol{r}$ is now performed in two dimensions.) The curious feature of (1.11) is that, in contrast to three-dimensional turbulence, it is almost universally assumed that $L$ is zero and that consequently $E \sim Ik^3$ for small $k$. (One of the notable exceptions to this is Lesieur & Herring 1985, who note that $E \sim k$ is indeed a possibility.) Certainly, nearly all numerical simulations show $E \sim k^3$, provided the computational domain is large enough (see, for example, Chasnov 1997; Ossai & Lesieur 2001; Lowe & Davidson 2005). It is natural to ask why this should be, and this provides one of the motivations for this paper. The same simulations also show the emergence of coherent vortices at late times, and it is natural to ask if there is a relationship between such vortices and the shape of $E(k)$.

The structure of the paper is as follows. In § 2 we lay the groundwork, introducing all the statistical quantities required for our discussion, as well as their relationship to each other. We also introduce the two-dimensional Kármán–Howarth equation and discuss some of its immediate consequences for $I$ and $L$, such as the fact that $L$ is an invariant. Although some of this material may be found scattered across various publications, we set it out systematically, if concisely, since it is not well documented elsewhere.

In § 3 we ask: are there any kinematic reasons why $E \sim k^3$ spectra are more natural than $E \sim Lk$ spectra? We shall see that there are not, and indeed we provide simple explicit examples of both $E \sim Lk$ and $E \sim Ik^3$ spectra. We shall see that, as in three dimension, the key point is whether or not the turbulent eddies possess finite linear impulse: if they do, then $E \sim Lk$, and if they do not, then $E \sim Ik^3$. We also show that homogeneous turbulence composed of a random sea of monopole vortices has a singular spectrum of the form $E \sim Jk^{-1}$, where the pre-factor $J$ is also an invariant. Note that the examples given in this section are purely kinematic.

We turn to dynamics in §§ 4–6. We start, in § 4, by considering $E \sim Lk$ turbulence. Here we show that the invariance of $L$ is a direct consequence of the principle of conservation of linear momentum. Now the fact that $L$ is an invariant means that a spectrum which starts out as $E \sim Lk$, must remain of this form. Lesieur & Herring (1985) note that such spectra cannot be self-similar at the large scales and they suggest that the $E \sim Lk$ part of the spectrum is progressively overshadowed by an increasingly strong $E \sim Ik^3$ region. We confirm this picture and give a simple argument which suggests that the wavenumber characteristic of the intersection of the $E \sim Lk$ and $E \sim Ik^3$ regions, $k^*$, decreases as $k^* \sim t^{-n}$, where $1 < n < 1.3$. Given that numerical simulations are usually performed in domains of modest size (relative to the integral scale), this suggests that the $E \sim Lk$ part of the spectrum will be rapidly lost.

We turn to $E \sim Jk^{-1}$ spectra in § 5, showing that the physical interpretation of the conservation of $J$ is similar to that of Corrsin's invariant in passive scalar mixing. In § 6, we consider the dynamics of $E \sim Ik^3$ turbulence, in which both $J$ and $L$ are set to zero by virtue of the initial conditions. Here we discuss the time-dependence of Loitsyansky's integral, $I$, and its relationship to the angular momentum of the turbulence.

We conclude, in § 7, with a brief discussion of the relationship between our findings and the vortical structures observed in numerical simulations.

## 2. Preliminaries

### 2.1. *The kinematics of two-dimensional turbulence*

The most general form of the two-point one-time velocity correlation, $Q_{ij} = \langle u_i u'_j \rangle$, in incompressible isotropic two-dimensional turbulence is,

$$Q_{ij}(\boldsymbol{r}) = u^2 \left\{ \frac{\partial}{\partial r}(rf)\delta_{ij} - \frac{r_i r_j}{r} f'(r) \right\}, \tag{2.1}$$

where $u^2 = \langle u_x^2 \rangle = \langle u_y^2 \rangle$, $r = |\boldsymbol{r}| = |\boldsymbol{x}' - \boldsymbol{x}|$, and $f(r)$ is the usual longitudinal correlation function. The triple velocity correlation tensor, $S_{ij\ell}(r) = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}) u_\ell(\boldsymbol{x} + \boldsymbol{r}) \rangle = \langle u_i u_j u'_l \rangle$, on the other hand, takes the form,

$$S_{ij\ell} = u^3 \left\{ \frac{r_i \delta_{j\ell} + r_j \delta_{i\ell}}{2r} \frac{\partial}{\partial r}(rK) - \frac{r_i r_j r_\ell}{r} \frac{\partial}{\partial r}\left(\frac{K}{r}\right) - \frac{r_\ell \delta_{ij} K}{r} \right\}, \tag{2.2}$$

where, as usual, $u^3 K = \langle u_x^2(\boldsymbol{x}) u_x(\boldsymbol{x} + r\hat{\boldsymbol{e}}_x) \rangle$ (see, for example, Davidson 2004, § 10.3). Two important special cases of (2.1) and (2.2) are

$$\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle = \frac{u^2}{r} \frac{\partial}{\partial r}(r^2 f) \tag{2.3}$$

and

$$S_{iji} = \frac{u^3 r_j}{2r^3} \frac{\partial}{\partial r}(r^3 K), \tag{2.4}$$

results to which we shall return shortly. The corresponding form of the spectral tensor,

$$\Phi_{ij}(\boldsymbol{k}) = \frac{1}{4\pi^2} \int Q_{ij} \exp(-j\boldsymbol{k} \cdot \boldsymbol{r}) \, \mathrm{d}\boldsymbol{r}, \tag{2.5}$$

is

$$\Phi_{ij}(\boldsymbol{k}) = \frac{E(k)}{\pi k} [\delta_{ij} - k_i k_j / k^2], \tag{2.6}$$

which is non-analytic at $k = 0$ for an $E(k \to 0) \sim Lk$ spectrum. Moreover, integrating over the polar angle in (2.5) shows that $E(k)$ and $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ are related via the Hankel transform pair,

$$E(k) = \tfrac{1}{2} \int_0^\infty \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle kr J_0(kr) \, \mathrm{d}r, \tag{2.7a}$$

$$\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle = 2 \int_0^\infty E(k) J_0(kr) \, \mathrm{d}k, \tag{2.7b}$$

provided, of course, that $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ decreases sufficiently rapidly with $|\boldsymbol{r}|$ for the integrals to exist. Expanding $J_0(kr)$ in a power series in $kr$ then yields

$$E(k) = Lk/4\pi + Ik^3/16\pi + \cdots, \tag{2.8}$$

where,

$$L = \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} = 2\pi u^2 [r^2 f]_\infty \tag{2.9}$$

and

$$I = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} \tag{2.10}$$

are the two-dimensional versions of the Saffman and Loitsyansky integrals, respectively. (Note that we have used (2.3) to relate $L$ to $f$ in (2.9).)

We shall see, in § 3.3, that $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$ does not always decay sufficiently rapidly with $|\boldsymbol{r}|$ for (2.7) to be meaningful. In particular, $E \sim k^{-1}$ spectra give rise to a logarithmic divergence of $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$. However, in such cases it turns out that the vorticity correlation, $\langle \omega\omega'\rangle$, is well behaved and so we may replace (2.7) by

$$E_\omega(k) = \tfrac{1}{2}\int_0^\infty \langle \omega\omega'\rangle kr\, J_0(kr)\,\mathrm{d}r, \tag{2.11}$$

$$\langle \omega\omega'\rangle = 2\int_0^\infty E_\omega(k) J_0(kr)\,\mathrm{d}k, \tag{2.12}$$

where

$$E_\omega(k) = k^2 E(k) \tag{2.13}$$

is the enstrophy spectrum. Again, expanding $J_0(kr)$ in a power series in $kr$ yields,

$$E(k) = \frac{J}{4\pi}k^{-1} + \frac{\hat{L}}{4\pi}k + \frac{\hat{I}}{16\pi}k^3 + \cdots, \tag{2.14}$$

where,

$$J = \int \langle \omega\omega'\rangle\,\mathrm{d}\boldsymbol{r}, \tag{2.15}$$

and

$$\hat{L} = -\tfrac{1}{4}\int r^2\langle \omega\omega'\rangle\,\mathrm{d}\boldsymbol{r}, \tag{2.16}$$

$$\hat{I} = \tfrac{1}{16}\int r^4\langle \omega\omega'\rangle\,\mathrm{d}\boldsymbol{r}. \tag{2.17}$$

The pre-factor $J$ can be expressed in terms of $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$ using

$$\langle \omega\omega'\rangle = -\nabla^2\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle, \tag{2.18}$$

from which,

$$J = -2\pi\left[r\frac{\partial}{\partial r}\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle\right]_\infty. \tag{2.19}$$

Evidently, the leading-order term in (2.14) vanishes when $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$ decays sufficiently rapidly with $|\boldsymbol{r}|$. Similarly, we may rewrite $\hat{L}$ and $\hat{I}$ in terms of $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$ using (2.18), and it is readily confirmed that, when $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty$ decays sufficiently rapidly with $|\boldsymbol{r}|$, $\hat{L} = L$ and $\hat{I} = I$. In particular,

$$\hat{L} = L \quad \text{for} \quad [r^2\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle]_\infty = 0, \qquad \hat{I} = I \quad \text{for} \quad [r^4\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle]_\infty = 0. \tag{2.20}$$

Thus we recover (1.11) and (2.8) in cases where $[r^4\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle]_\infty = 0$.

### 2.2. *The Kármán–Howarth equation in two dimensions*

We now move from kinematics to dynamics. As in three dimensions, the Navier–Stokes equation provides the evolution equation

$$\frac{\partial Q_{ij}}{\partial t} = \frac{\partial}{\partial r_\ell}[S_{j\ell i} + S_{i\ell j}] + 2\nu\nabla^2 Q_{ij}. \tag{2.21}$$

Setting $i = j$, and using (2.4) to evaluate $S_{iji}$, this yields the two-dimensional Kármán–Howarth equation:

$$\frac{\partial}{\partial t}\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle = \frac{u^3}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r^3 K) + 2\nu\nabla^2\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle. \tag{2.22}$$

Evolution equations for $I$ and $L$ follow directly from (2.22). They are,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 2\pi u^3 \left[\frac{1}{r}\frac{\partial}{\partial r}(r^3 K)\right]_\infty, \tag{2.23}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\pi u^3 \left[2r^3 K - r\frac{\partial}{\partial r}(r^3 K)\right]_\infty - 8\nu L, \tag{2.24}$$

where we have integrated by parts to obtain (2.24). Note that we have dropped certain viscous terms in (2.23) and (2.24) on the assumption that $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty$ decays as $r^{-3}$, or faster. (We shall confirm shortly that this is valid for $E \sim k$ and $E \sim k^3$ spectra, but not, of course, $E \sim k^{-1}$ spectra.)

There now arises the question of the asymptotic form of $K_\infty$. This, in turn, is controlled by the long-range pressure forces, as described by Batchelor & Proudman (1956) for three-dimensional turbulence. A reworking of their analysis in two dimensions yields $K_\infty \sim r^{-3}$ (Davidson 2004, § 10.3). That is, a localized vortex patch sets up far-field pressure fluctuations which fall off as $p_\infty \sim r^{-2}$ and this, in turn, produces velocity–pressure correlations of the form $\langle u_i u_j p'\rangle_\infty \sim r^{-2}$. The gradients of just such correlations appear as source terms in the evolution equation for $S_{ijk}$, leading to $K_\infty \sim r^{-3}$. It follows that (2.23) and (2.24) reduce to

$$L = \text{constant}, \tag{2.25}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 4\pi u^3 [r^3 K]_\infty - 8\nu L. \tag{2.26}$$

These results can also be reached via a standard spectral analysis (see, for example, Lesieur & Herring 1985, or Davidson 2004, § 10.3). The easiest way is to take the Hankel transform of (2.22) in accordance with (2.7). Noting that $K_\infty \sim r^{-3}$, or faster, we obtain,

$$\frac{\partial E}{\partial t} = k^3 \int_0^\infty \frac{\partial}{\partial r}[r^3 u^3 K]\frac{J_1(kr)}{2kr}\,\mathrm{d}r - 2\nu k^2 E.$$

When combined with expansion (2.8), we recover (2.25) and (2.26).

In cases where $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty$ is not well behaved, but $\langle\omega\omega'\rangle_\infty$ is, such as $E \sim k^{-1}$ spectra, we can find analogous equations for $J$, $\hat{L}$ and $\hat{I}$. The procedure is straightforward. The Laplacian of (2.22) yields

$$\frac{\partial}{\partial t}\langle\omega\omega'\rangle = -\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r^3 u^3 K)\right] + 2\nu\nabla^2\langle\omega\omega'\rangle, \tag{2.27}$$

the integral moments of which provide evolution equations for $J$, $\hat{L}$ and $\hat{I}$. These equations involve integrals of derivatives of $K$ and $\langle\omega\omega'\rangle$ on the right-hand side. Integrating these by parts, and noting that $K_\infty \sim r^{-3}$, we find, after a little algebra

$$J = \text{constant}, \tag{2.28}$$

$$\frac{\mathrm{d}\hat{L}}{\mathrm{d}t} = -2\nu J, \tag{2.29}$$

$$\frac{\mathrm{d}\hat{I}}{\mathrm{d}t} = 4\pi u^3 [r^3 K]_\infty - 8\nu\hat{L}. \tag{2.30}$$

(In order to simplify the viscous terms in (2.28)–(2.30) we have assumed that $\langle\omega\omega'\rangle_\infty$ decays faster than $r^{-4}$, which will be justified shortly.) Thus, in the limit of large $Re$, $J$ and $\hat{L}$ are invariants. Of course, (2.28)–(2.30) are equivalent to (2.25) and (2.26) for cases in which $[r^4\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle]_\infty = 0$.

One other consequence of $K_\infty \sim r^{-3} + O(r^{-4})$ is that (2.22) demands that $\partial\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty/\partial t \sim r^{-5}$. Hence $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty \sim r^{-5}$, unless $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty$ happens to fall off more slowly at $t=0$. We shall see that $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty \sim r^{-5}$ is typical of $E\sim k$ and $E\sim k^3$ spectra, justifying the neglect of certain viscous terms in (2.23) and (2.24). However, for $E\sim k^{-1}$ spectra, (2.19) tells us that $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty$ diverges logarithmically at $t=0$, and since $J$ is an invariant, this situation persists. In such cases we must turn to (2.27), which tells us that $\partial\langle\omega\omega'\rangle_\infty/\partial t \sim r^{-7}$, and hence $\langle\omega\omega'\rangle_\infty \sim r^{-7}$, unless $\langle\omega\omega'\rangle_\infty$ falls off more slowly at $t=0$. We shall see that $\langle\omega\omega'\rangle_\infty \sim r^{-7}$ is typical of $E\sim k^{-1}$ spectra.

Returning to $E\sim k$ and $E\sim k^3$ spectra, the estimate $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty \sim r^{-5}$, combined with $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle = -\nabla^2\langle\psi\psi'\rangle$, tells us that $\langle\psi\psi'\rangle_\infty \sim r^{-3}$, where $\psi$ is the streamfunction. It follows that $I$ can be rewritten in terms of $\langle\psi\psi'\rangle$ in the form:

$$I = 4\int \langle\psi\psi'\rangle\,\mathrm{d}\boldsymbol{r}. \tag{2.31}$$

We shall make use of (2.31) later.

Note that, so far, there is nothing in our analysis which would rule out either $E\sim Lk$, or $E\sim Jk^{-1}$, spectra. Indeed, since

$$L = \left\langle \left[\int_V \boldsymbol{u}\,\mathrm{d}V\right]^2 \right\rangle \Big/ V, \tag{2.32}$$

where $V$ is some large two-dimensional volume, we might expect that $L$ is non-zero whenever the turbulence has sufficient linear momentum. Moreover, the linear momentum of a two-dimensional flow is proportional to the sum of the linear impulses, $\int \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V$, of the eddies (filaments or blobs of vorticity) within the flow. So a loose application of the central limit theorem, along the lines of § 1.1, suggests that $L$ will be non-zero if, on average, the eddies have a finite linear impulse. We shall now show that this is indeed the case.

## 3. Simple kinematic examples of $E\sim Ik^3$, $E\sim Lk$ and $E\sim Jk^{-1}$ spectra

### 3.1. *An example of an $E\sim Ik^3$ spectrum*

Consider a simple eddy, described in polar coordinates as

$$\boldsymbol{u} = \Omega r \exp[-2r^2/\ell^2]\hat{\boldsymbol{e}}_\theta, \tag{3.1}$$

where $\Omega$ and $\ell$ are constants. This has zero linear impulse, $\int \boldsymbol{x}\times\boldsymbol{\omega}\mathrm{d}V=0$, but finite angular impulse and angular momentum. Clearly, its streamfunction is

$$\psi = \psi_0 \exp[-2r^2/\ell^2], \tag{3.2}$$

where $\psi_0 = \Omega\ell^2/4$. Now suppose that such eddies are randomly but uniformly distributed in space to form an artificial field of turbulence composed of eddies of fixed size $\ell$. If the sign of $\psi_0$ for each eddy is also randomly chosen, subject to the constraint that $\langle\psi_0\rangle = 0$, then we may write

$$\psi = \sum_n \delta_n|\psi_0|\exp[-2(\boldsymbol{x}-\boldsymbol{x}_n)^2/\ell^2], \tag{3.3}$$

where $\delta_n = \pm 1$ and $\boldsymbol{x}_n$ locates the $n$th eddy. Thus, $\delta_n$ and $\boldsymbol{x}_n$ constitute a set of independent random variables, with $\langle\delta_n\rangle = 0$ and $\boldsymbol{x}_n$ uniformly distributed in space.

We now form the two-point correlation,

$$\langle \psi(0)\psi(\boldsymbol{r})\rangle = \sum_m \sum_n \langle \delta_m \delta_n \rangle \psi_0^2 \langle \exp\left[-2\left(\boldsymbol{x}_m^2 + (\boldsymbol{r} - \boldsymbol{x}_n)^2\right)/\ell^2\right]\rangle, \tag{3.4}$$

which we can rewrite as

$$\langle \psi\psi' \rangle = \psi_0^2 \sum_n \langle \exp\left[-2\left(\boldsymbol{x}_n^2 + (\boldsymbol{r} - \boldsymbol{x}_n)^2\right)/\ell^2\right]\rangle, \tag{3.5}$$

since $\langle \delta_n \delta_m \rangle = \delta_{nm}$. This may be rearranged into the form

$$\langle \psi\psi' \rangle = \psi_0^2 \exp(-r^2/\ell^2) \sum_n \langle \exp\left[-4\boldsymbol{y}_n^2/\ell^2\right]\rangle, \tag{3.6}$$

where $\boldsymbol{y}_n = \boldsymbol{x}_n - \boldsymbol{r}/2$ is a new random variable, obtained from $\boldsymbol{x}_n$ by a shift of origin. It follows that, for this artificial field of turbulence,

$$\langle \psi\psi' \rangle = \langle \psi^2 \rangle \exp(-r^2/\ell^2). \tag{3.7}$$

Moreover, since

$$\langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle = -\nabla^2 \langle \psi\psi' \rangle = \frac{u^2}{r}\frac{\partial}{\partial r}(r^2 f), \tag{3.8}$$

we obtain,

$$f = \exp(-r^2/\ell^2), \quad \langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle = 2u^2[1 - r^2/\ell^2]\exp(-r^2/\ell^2), \tag{3.9}$$

and hence, from (2.7),

$$E(k) = \tfrac{1}{2}\langle \boldsymbol{u}^2 \rangle \ell (k\ell/2)^3 \exp[-k^2\ell^2/4]. \tag{3.10}$$

An expression similar to (3.9) is stated without proof in Townsend (1976), although Townsend constructed his field of turbulence in a different way. In any event, the key result in the present context is that we have $L = 0$ and $E \sim k^3$. Of course, this is precisely what we expect, since our model eddy has zero net linear impulse.

Now suppose that, instead of having eddies of just one size, we have a continuous distribution of sizes. Let $s$ represent eddy size, and $\hat{E}(s)$ be the energy density, defined by the fact that $\hat{E}(s)\,\mathrm{d}s$ gives the contribution to $\langle \boldsymbol{u}^2 \rangle/2$ which comes from eddies in the size range $s \to s + \mathrm{d}s$, i.e.

$$\tfrac{1}{2}\langle \boldsymbol{u}^2 \rangle = \int_0^\infty \hat{E}(s)\,\mathrm{d}s = \int_0^\infty E(k)\,\mathrm{d}k.$$

If all of the eddies are of type (3.1), but of varying size, then the principle of superposition allows us to generalize (3.10) to give,

$$E(k) = \int_0^\infty \hat{E}(s)s(ks/2)^3 \exp[-k^2s^2/4]\,\mathrm{d}s.$$

Of course, like (3.10), this is of the form $E \sim k^3$ since the flow has no linear impulse. Let us now repeat the analysis, but with a model eddy that does have some linear impulse.

### 3.2. *An example of an $E \sim Lk$ spectrum*

Consider a model eddy whose vorticity field takes the form

$$\omega = \Omega(x/\ell)\exp[-2(x^2 + y^2)/\ell^2], \tag{3.11}$$

where $\Omega$ and $\ell$ are, once again, constants. Clearly, this is a dipole field with non-zero linear impulse. If such eddies are randomly but uniformly distributed in space, and with random orientation and sign, then the net vorticity can be written as

$$\omega = \sum_n \delta_n |\Omega| ((\boldsymbol{x} - \boldsymbol{x}_n) \cdot \hat{\boldsymbol{e}}_n / \ell) \exp[-2(\boldsymbol{x} - \boldsymbol{x}_n)^2 / \ell^2]. \tag{3.12}$$

Here $\delta_n = \pm 1$, $\boldsymbol{x}_n$ locates the $n$th eddy, and $\hat{\boldsymbol{e}}_n$ is a unit vector which determines the orientation of the $n$th eddy. Thus, $\delta_n$, $\boldsymbol{x}_n$ and $\hat{\boldsymbol{e}}_n$ constitute a set of independent random variables, with $\langle \delta_n \rangle = 0$, $\langle (\hat{\boldsymbol{e}}_n)^2 \rangle = 1$, and $\boldsymbol{x}_n$ uniformly distributed in space. We now form the two-point vorticity correlation

$$\langle \omega(0)\omega(\boldsymbol{r}) \rangle = \sum_m \sum_n \langle \delta_m \delta_n \rangle (\Omega/\ell)^2 \big\langle (\boldsymbol{x}_m \cdot \hat{\boldsymbol{e}}_m)((\boldsymbol{x}_n - \boldsymbol{r}) \cdot \hat{\boldsymbol{e}}_n) \exp\big[-2\big(\boldsymbol{x}_m^2 + (\boldsymbol{r} - \boldsymbol{x}_n)^2\big)/\ell^2\big] \big\rangle,$$

and since $\langle \delta_n \delta_m \rangle = \delta_{nm}$, this simplifies to

$$\langle \omega\omega' \rangle = (\Omega/\ell)^2 \sum_n \big\langle (\boldsymbol{x}_n \cdot \hat{\boldsymbol{e}}_n)((\boldsymbol{x}_n - \boldsymbol{r}) \cdot \hat{\boldsymbol{e}}_n) \exp\big[-2\big(\boldsymbol{x}_n^2 + (\boldsymbol{r} - \boldsymbol{x}_n)^2\big)/\ell^2\big] \big\rangle. \tag{3.13}$$

As before, we replace $\boldsymbol{x}_n$ by the new random variable $\boldsymbol{y}_n = \boldsymbol{x}_n - \boldsymbol{r}/2$, which is obtained from $\boldsymbol{x}_n$ by a shift of origin. In terms of $\boldsymbol{y}_n$, (3.13) becomes

$$\langle \omega\omega' \rangle = (\Omega/\ell)^2 \exp(-r^2/\ell^2) \sum_n \big\langle \big[(\boldsymbol{y}_n \cdot \hat{\boldsymbol{e}}_n)^2 - \tfrac{1}{4}(\boldsymbol{r} \cdot \hat{\boldsymbol{e}}_n)^2\big] \exp\big[-4\boldsymbol{y}_n^2/\ell^2\big] \big\rangle \tag{3.14}$$

and since $\hat{\boldsymbol{e}}_n$ and $\boldsymbol{y}_n$ are statistically independent, this yields,

$$\langle \omega\omega' \rangle = (\Omega/\ell)^2 \exp(-r^2/\ell^2) \sum_n \big[\tfrac{1}{2}\big\langle \boldsymbol{y}_n^2 \exp\big[-4\boldsymbol{y}_n^2/\ell^2\big]\big\rangle - \tfrac{1}{8} r^2 \big\langle \exp\big[-4\boldsymbol{y}_n^2/\ell^2\big]\big\rangle\big]. \tag{3.15}$$

The final step is to rewrite (3.15) in terms of the dimensionless random variable $z_n^2 = 4\boldsymbol{y}_n^2/\ell^2$:

$$\langle \omega\omega' \rangle = \tfrac{1}{8}\Omega^2 \exp(-r^2/\ell^2) \textstyle\sum_n \big[\big\langle z_n^2 \exp\big(-z_n^2\big)\big\rangle - (r/\ell)^2 \big\langle \exp\big(-z_n^2\big)\big\rangle\big]. \tag{3.16}$$

Now $\boldsymbol{x}_n$, and hence $\boldsymbol{y}_n$, are uniformly distributed in space, and so it follows that $\langle z_n^2 \exp(-z_n^2) \rangle = \langle \exp(-z_n^2) \rangle$, since

$$\int_0^\infty z^2 \exp(-z^2)\, \mathrm{d}z^2 = \int_0^\infty \exp(-z^2)\, \mathrm{d}z^2 = 1.$$

We conclude, therefore, that the two-point vorticity correlation is simply,

$$\langle \omega\omega' \rangle = \langle \omega^2 \rangle [1 - r^2/\ell^2] \exp(-r^2/\ell^2). \tag{3.17}$$

Expressions for $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ and $f(r)$ follow immediately from $\langle \omega\omega' \rangle = -\nabla^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ and (2.3). We find,

$$\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle = \langle \boldsymbol{u}^2 \rangle \exp(-r^2/\ell^2) \tag{3.18}$$

and

$$f(r) = \frac{1 - \exp(-r^2/\ell^2)}{(r/\ell)^2}. \tag{3.19}$$

It is interesting to compare (3.18) and (3.19) with (3.9). In both cases, $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ decays exponentially, so that $I$ is well defined and expansion (2.8) justified. However, the crucial difference is that $f_\infty$ is exponentially small in (3.9), so that $L = 0$, whereas $f_\infty \sim r^{-2}$ in (3.19), which yields a non-zero value of $L$ in accordance with (2.9). In fact, (2.7) gives us

$$E(k) = \tfrac{1}{2}\langle \boldsymbol{u}^2 \rangle \ell [k\ell/2] \exp[-k^2\ell^2/4]. \tag{3.20}$$

Thus, as anticipated in § 2.2, a random sea of dipoles yields a non-zero $L$ and hence an $E \sim Lk$ spectrum. The analysis is readily generalized to a sea of dipole eddies with a continuous distribution of sizes, $s$, as in § 3.1.

### 3.3. *An example of a singular, $E \sim Jk^{-1}$, spectrum*

We close this section with an example where $E(k)$ is divergent. Suppose that our artificial field of turbulence is now composed of a random sea of monopole vortices of arbitrary sign, of the form $\omega = \omega_0 \exp[-2r^2/\ell^2]$. Repeating the analysis of § 3.1, but with $\omega$ replacing $\psi$, yields, by virtue of (3.7),

$$\langle \omega\omega' \rangle = \langle \omega^2 \rangle \exp(-r^2/\ell^2). \tag{3.21}$$

Next, using (2.11), we obtain a vorticity power spectrum, $E_\omega(k)$, of the form,

$$E_\omega(k) = k^2 E(k) = \tfrac{1}{2}\langle \omega^2 \rangle \ell [k\ell/2] \exp[-k^2\ell^2/4]. \tag{3.22}$$

The corresponding energy spectrum is therefore divergent for small $k$, $E \sim k^{-1}$, reflecting the fact that the energy of a monopole vortex, or indeed a homogeneous random sea of such vortices, is divergent. (Note, however, that the divergence of $\langle \boldsymbol{u}^2 \rangle \sim \int E\,\mathrm{d}k$ caused by the existence of monopoles is avoided in numerical simulations in a periodic square, since there is a lower cutoff in $k$.)

The form of $\langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle$ corresponding to spectrum (3.22) can be found from (2.18). One integration yields

$$r\frac{\partial}{\partial r}\langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle = -\tfrac{1}{2}\langle \omega^2 \rangle \ell^2 [1 - \exp(-r^2/\ell^2)], \tag{3.23}$$

from which,

$$\langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle - \langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle_{r=a} = \tfrac{1}{4}\langle \omega^2 \rangle \ell^2 [\mathrm{Ein}(a^2/\ell^2) - \mathrm{Ein}(r^2/\ell^2)], \tag{3.24}$$

where $\mathrm{Ein}(x)$ is the usual exponential integral. Note that, for large $x$, $\mathrm{Ein}(x) \sim \ln(x)$, and so $\langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle$ diverges logarithmically, as noted in § 2.1.

Now suppose that, instead of having eddies of just one size, we have a continuous distribution of sizes. Let $s$ represent eddy size, and $\hat{E}_\omega(s)$ be the enstrophy density, defined by

$$\tfrac{1}{2}\langle \omega^2 \rangle = \int_0^\infty \hat{E}_\omega(s)\,\mathrm{d}s = \int_0^\infty E_\omega(k)\,\mathrm{d}k.$$

If all of the eddies are monopoles, of the form $\omega = \omega_0 \exp[-2r^2/s^2]$, but of varying size, then, as in § 3.1, we may repeat the analysis using superposition to give,

$$E_\omega(k) = \int_0^\infty \hat{E}_\omega(s) s [ks/2] \exp[-k^2 s^2/4]\,\mathrm{d}s.$$

Of course, like (3.22), this is of the form $E_\omega \sim k$ at small $k$, corresponding to the singular energy spectrum $E \sim k^{-1}$.

We close this section with a note of warning. While we have shown that a sea of randomly located monopoles gives rise to $E \sim k^{-1}$ spectrum, the same need not be true of a sea of monopoles whose spatial locations are somehow constrained. We shall return to this in § 7, where we shall see that the constraint imposed by the conservation of energy rules out an $E \sim k^{-1}$ spectrum for monopoles which have emerged from certain types of initial conditions.

### 3.4. *A summary of the kinematic results*

Perhaps it is worth summarizing the results of §§3.1–3.3. We have shown that an artificial field of turbulence composed of eddies with zero linear impulse, but finite angular impulse, has $L = 0$, and hence $E \sim Ik^3$, whereas a random distribution of eddies with finite linear impulse has $L \neq 0$, and hence $E \sim Lk$. A random cloud of monopoles, on the other hand, has $E \sim Jk^{-1}$. In summary, then, we have:

turbulence composed of random monopoles: $E(k \to 0) = \dfrac{J}{4\pi}k^{-1}, \quad J = \displaystyle\int \langle \omega\omega' \rangle \, \mathrm{d}\boldsymbol{r};$

turbulence composed of random dipoles: $E(k \to 0) = \dfrac{L}{4\pi}k, \quad L = \displaystyle\int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r};$

turbulence with $J = L = 0$ : $E(k \to 0) = \dfrac{I}{16\pi}k^3, \quad I = -\displaystyle\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r}.$

We shall see, in §6, that $I$ is related to the angular momentum of the turbulence. In principle, then, we could have kinematically admissible fields of turbulence in which $J = L = I = 0$, and these would have $E \sim k^5$ spectra, or perhaps some higher exponent. Such fields are readily generated by choosing a model eddy with zero linear and angular impulse and then repeating the analysis of §3.1. However, the transient growth of $I(t)$ in two-dimensional turbulence, discussed in §6, means that turbulence which starts out as $E \sim k^5$ will rapidly develop an $E \sim Ik^3$ component at small $k$.

## 4. The large-scale dynamics of $E \sim Lk$ turbulence

The analysis of §3 is purely kinematic, so we now turn to dynamics. In this section, we restrict ourselves to $E \sim Lk$ spectra, leaving the cases of $E \sim Jk^{-1}$ and $E \sim Ik^3$ until §§5 and 6, respectively. There are two main points we wish to explore here. First, we shall show that the invariance of $L$, noted in §2.2, is an immediate consequence of the principle of conservation of linear momentum. Secondly, Lesieur & Herring (1985) have suggested that $E \sim Lk$ turbulence cannot evolve in a self-similar manner at the large scales. Indeed, they suggest that the $E \sim Lk$ part of the spectrum will be rapidly overshadowed by the transient growth of an $E \sim Ik^3$ component. We shall confirm this. First, however, let us start with the physical interpretation of (2.25).

### 4.1. *The physical interpretation of the invariance of L*

Consider a circular control surface, $S$, of radius $R$, embedded in a sea of homogeneous turbulence. We shall apply the principle of conservation of momentum to the two-dimensional volume, $V$, enclosed by $S$, and show that this leads directly to the invariant (2.25). That is to say, we shall show that $L$ is a measure of the square of the linear momentum held within $V$, and that this is conserved in the limit of large $R$ because the flux of momentum across the surface $S$ is negligible. In the process, we provide independent confirmation of (2.32):

$$L = \left\langle \left[ \int_V \boldsymbol{u} \, \mathrm{d}V \right]^2 \right\rangle \Big/ V.$$

The first step is to evaluate the total linear momentum, $\boldsymbol{L} = \int \boldsymbol{u} \mathrm{d}V$, contained within $V$. (Actually, it turns out to be more convenient to work with $\boldsymbol{L}^2$, for reasons that will become apparent.) Noting that $u_i = \nabla \cdot (\boldsymbol{u} x_i)$, we may write

$$\boldsymbol{L}^2 = \int_V \boldsymbol{u} \, \mathrm{d}V \cdot \oint_S \boldsymbol{x}(\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}), \tag{4.1}$$

which yields

$$\langle \boldsymbol{L}^2 \rangle = \left\langle \int_V \left[ u_i \oint_S x_i \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} \right] \mathrm{d}V \right\rangle . \tag{4.2}$$

Let us take the origin for $\boldsymbol{x}$ to lie at the centre of $V$. Since all points on $S$ are statistically equivalent, we focus on the boundary point $\boldsymbol{x} = -R\hat{\boldsymbol{e}}_x$ when evaluating the inner integral. Then (4.2) simplifies to

$$\langle \boldsymbol{L}^2 \rangle = 2\pi R^2 \int_V \langle u_x u'_x \rangle \, \mathrm{d}\boldsymbol{r}, \tag{4.3}$$

where $u_x$ is evaluated on the surface, at $\boldsymbol{x} = -R\hat{\boldsymbol{e}}_x$, and $\boldsymbol{x}'$ is an interior point within $V$. Thus the displacement vector $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$ links $\boldsymbol{x} = -R\hat{\boldsymbol{e}}_x$ to the interior point $\boldsymbol{x}'$. Next, substituting for $\langle u_x u'_x \rangle$, using (2.1), we find

$$\langle \boldsymbol{L}^2 \rangle = 2\pi R^2 u^2 \int_V \left[ \frac{\partial}{\partial r}(rf) - \frac{r_x^2}{r} f'(r) \right] \mathrm{d}\boldsymbol{r}. \tag{4.4}$$

Since the integration is now over $\boldsymbol{r}$, it is convenient to take a new origin at the boundary point $\boldsymbol{x} = -R\hat{\boldsymbol{e}}_x$, and introduce the polar angle $\phi$, defined by $\cos\phi = \boldsymbol{r} \cdot \hat{\boldsymbol{e}}_y / r$. In terms of $r$ and $\phi$, we have

$$\langle \boldsymbol{L}^2 \rangle = 4\pi R^2 u^2 \int_0^{2R} \int_{\hat{\phi}}^{\pi/2} \left[ r \frac{\partial}{\partial r}(rf) - f'(r) r^2 \sin^2\phi \right] \mathrm{d}\phi \, \mathrm{d}r, \tag{4.5}$$

where $\hat{\phi} = \sin^{-1}(r/2R)$. Integration over $\phi$ is now straightforward and this yields,

$$\frac{\langle \boldsymbol{L}^2 \rangle}{\pi R^2} = \frac{4u^2}{R} \int_0^{2R} [1 - (r/2R)^2]^{1/2} r^2 f(r) \, \mathrm{d}r. \tag{4.6}$$

We shall return to (4.6) shortly.

Let us now apply the principle of conservation of linear momentum to our control volume. Ignoring viscous forces we have

$$\frac{\mathrm{d}L_i}{\mathrm{d}t} = -\oint_S u_i \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} - \oint_S (p/\rho) \, \mathrm{d}S_i, \tag{4.7}$$

from which

$$\frac{\mathrm{d}\boldsymbol{L}^2}{\mathrm{d}t} = -2 \int_V u_i \, \mathrm{d}V \left[ \oint_S u_i \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} + \oint_S (p/\rho) \, \mathrm{d}S_i \right] . \tag{4.8}$$

Since $\langle \boldsymbol{u} p' \rangle = 0$ in isotropic turbulence, the pressure term vanishes on averaging and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{L}^2 \rangle = -2 \left\langle \int_V \left[ u_i \oint_S u_i \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} \right] \mathrm{d}V \right\rangle , \tag{4.9}$$

which is reminiscent of (4.2). As before, we note that all points on the boundary are statistically equivalent, and fixing on the surface point $\boldsymbol{x} = -R\hat{\boldsymbol{e}}_x$, (4.9) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{L}^2 \rangle = 4\pi R \int_V \langle u_i u_x u'_i \rangle \, \mathrm{d}\boldsymbol{r}, \tag{4.10}$$

where $u_i u_x$ is evaluated on the boundary, at $\boldsymbol{x} = -R\hat{\boldsymbol{e}}_x$, and $\boldsymbol{x}'$ is an interior point within $V$. Equation (4.10) can be evaluated in exactly the same way as (4.3), and we find, after a little algebra,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\langle \boldsymbol{L}^2\rangle/\pi R^2] = \frac{4}{R}\int_0^{2R}[1-(r/2R)^2]^{1/2}\frac{1}{r}\frac{\partial}{\partial r}(r^3u^3K)\,\mathrm{d}r. \tag{4.11}$$

Let us now interpret (4.6) and (4.11). Comparing (4.6) with (2.32) we see that $\langle \boldsymbol{L}^2\rangle/\pi R^2$ should, in the limit of $R\rightarrow\infty$, yield $L$. This is readily confirmed since the right-hand side of (4.6) tends to $2\pi u^2[r^2 f]_\infty$ as $R\rightarrow\infty$, consistent with (2.9). In short, (4.6) provides independent confirmation that

$$L = \operatorname*{Lim}_{R\rightarrow\infty}\frac{\langle \boldsymbol{L}^2\rangle}{\pi R^2}. \tag{4.12}$$

Let us now consider the more important result, (4.11). Using (4.12), this may be rewritten as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \operatorname*{Lim}_{R\rightarrow\infty}\frac{4}{R}\int_0^{2R}[1-(r/2R)^2]^{1/2}\frac{1}{r}\frac{\partial}{\partial r}(r^3u^3K)\,\mathrm{d}r, \tag{4.13}$$

and it is readily confirmed that the right-hand side of this equation is indeed zero when $K_\infty\sim r^{-3}$. Now for a finite but large value of $R$, the left-hand side of (4.13) is proportional to the rate of change of $\boldsymbol{L}^2$, the square of the linear momentum in $V$. The right-hand side, on the other hand, is proportional to the flux of linear momentum out through the control surface $S$. This provides us with a simple physical interpretation of the conservation of $L$: for large but finite $R$, $L$ is a measure of $\boldsymbol{L}^2$, and this is conserved because there is negligible flux of linear momentum out through the control surface $S$. In effect, conservation of $L$ follows directly from the conservation of linear momentum.

### 4.2. *The implications of the conservation of L*

In the limit of $\nu\rightarrow 0$, kinetic energy is conserved in two-dimensional turbulence. Thus, for a small but finite viscosity, we have at least two invariants, $u^2$ and $L$. On the other hand, it is well known that the integral scale, defined, say, as $\ell = u^2/\int_0^\infty kE\,\mathrm{d}k$, continually grows in freely decaying turbulence (see, for example, Davidson 2004, § 10.1.4). The precise behaviour of $\ell$ depends on how it is defined, but typically it grows as $\ell\sim\sqrt{t}$ (see Lowe & Davidson 2005, and references therein). As noted in Lesieur & Herring (1985), this tells us that the large scales cannot evolve in a self-similar fashion in $E\sim Lk$ turbulence. That is, if the large scales were self-similar, then

$$L = \int\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r} = Cu^2\ell^2,$$

where the pre-factor, $C$, is independent of time; but this is not possible if $u^2$ is conserved as $\ell$ grows. Thus, we conclude that the small-$k$ end of the dimensionless energy spectrum, $E(k\ell)/u^2\ell$, changes shape as the turbulence evolves. This is in marked contrast to three-dimensional turbulence where the large scales are observed to be approximately self-similar. Closure estimates, using the test-field model, suggests that this non-self-similar evolution consists of the $E\sim Lk$ part of the spectrum being progressively overshadowed by the transient growth of an $E\sim Ik^3$ component (Lesieur & Herring 1985), and this is consistent with the large-eddy simulations of Ossai & Lesieur (2001). This behaviour can be understood as follows. We have

$$E(k) = Lk/4\pi + Ik^3/16\pi + \cdots,$$

and so the wavenumber characteristic of the intersection of the $E \sim Lk$ and $E \sim Ik^3$ regions, $k^*$, scales as $k^* \sim \sqrt{L/I}$. However, $L$ is an invariant while $I$ grows as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 4\pi u^3 [r^3 K]_\infty.$$

Estimates of $\mathrm{d}I/\mathrm{d}t$ obtained from numerical simulations vary, but typically $I \sim t^{2.3\pm0.3}$ (see for example, Ossai & Lesieur 2001), which yields $k^* \sim t^{-n}$, $1 < n < 1.3$. This suggests that the $E \sim Lk$ part of the spectrum is indeed progressively overshadowed by the $E \sim Ik^3$ component, even when the spectrum is represented in normalized form, such as $E(k\ell)/u^2\ell$.

By way of a simple example, consider the somewhat artificial case where

$$E(k) = c_1 k + c_3 k^3, \quad k < \hat{k}, \tag{4.14}$$

and $E \approx 0$ for $k > \hat{k}$. Then,

$$c_3 = I/16\pi = 2[\langle \boldsymbol{u}^2 \rangle - c_1 \hat{k}^2]\hat{k}^{-4}, \tag{4.15}$$

while $c_1$ is conserved by virtue of (2.25). We now take $c_3(0) = 0$ and $\hat{k} = \hat{k}_0(1+t/\tau)^{-1/2}$, in line with $\ell \sim \sqrt{t}$, $\tau$ being the initial eddy turnover time. Conservation of energy then tells us that, for large $t/\tau$, $I$ grows as $I \sim t^2$, though the rate of rise will be somewhat faster at moderate times. Moreover, the transition from linear to cubic behaviour in $E$ occurs at around

$$k^* = (c_1/c_3)^{1/2} = \frac{\sqrt{c_1/2}\,\hat{k}^2}{[\langle \boldsymbol{u}^2 \rangle - c_1 \hat{k}^2]^{1/2}}, \quad t/\tau \geqslant 1/2, \tag{4.16}$$

which scales as $k^* \sim t^{-1}$ for $t \gg \tau$. Thus, both $\hat{k}$ and $k^*$ decrease with time, but $k^*$ decreases faster.

## 5. The large-scale dynamics of $E \sim Jk^{-1}$ turbulence

Let us now turn to other types of spectra, starting with the curious case of $E \sim Jk^{-1}$. We have seen that an artificial field of turbulence composed of a random cloud of monopoles of arbitrary sign yields a singular spectrum of the form $E(k \to 0) = (J/4\pi)k^{-1}$. For strictly isotropic turbulence, this leads to the pathological result that $\langle \boldsymbol{u}^2 \rangle$ diverges, and so it is tempting to dismiss this case out of hand. However, it should be remembered that numerical simulations in a periodic square have a lower cutoff in $k$, and so $E \sim Jk^{-1}$ type spectra can be observed with finite energy. Thus, the case of $E \sim k^{-1}$ is of some interest.

It is natural to ask if the pre-factor $J$ has a simple physical interpretation, like that of $L$. In § 2.1, we saw that

$$J = \int \langle \omega\omega' \rangle \, \mathrm{d}\boldsymbol{r}, \tag{5.1}$$

which, in turn, yields

$$J = \lim_{V\to\infty} \left\langle \left[ \int_V \omega \, \mathrm{d}V \right]^2 \right\rangle \bigg/ V. \tag{5.2}$$

That is, $J$ is a measure of the net vorticity contained in a large two-dimensional volume, $V$. Now recall that $J$ is non-zero when the large-scale vortices consist of a collection of randomly located monopoles. In such a case, (5.2) tells us that we obtain incomplete cancellation of vorticity within a finite volume $V$, with $\int_V \omega \, \mathrm{d}V \sim V^{1/2}$.

This incomplete cancellation of vorticity within any finite volume is why the energy density of a homogeneous sea of monopoles is divergent, despite the fact that $\langle\omega\rangle = 0$.

A physical interpretation of the conservation of $J$ may be obtained from a consideration of (5.2). However, perhaps this is most readily understood by considering first the analogous problem in passive scalar mixing. In the passive scalar problem, the contaminant field, $c(\boldsymbol{x}, t)$, has the invariant $J_c = \int \langle cc'\rangle \,\mathrm{d}\boldsymbol{r}$, known as Corrsin's integral. The conservation of $J_c$ can be demonstrated by integrating the evolution equation for $\langle cc'\rangle$ over all $\boldsymbol{r}$, while noting that those statistical correlations whose origins lie in the advection and diffusion of $c$ take the form of divergences, which integrate to zero. So, physically, $J_c$ is conserved because: (i) it can be written in the form,

$$J_c = \lim_{V\to\infty} \left\langle \left[\int_V c\mathrm{d}V\right]^2 \right\rangle \Big/ V; \tag{5.3}$$

and (ii) $\int_V c\,\mathrm{d}V$ is itself conserved since the advection and diffusion of $c$ across the surface of $V$ is too weak to change this integral in the limit of $V \to \infty$.

The same argument can be applied to (5.2) in order to explain the conservation of $J$. That is, in the limit of large $V$, $\int_V \omega\,\mathrm{d}V$ is conserved because the advection and diffusion of vorticity across the surface of $V$ is too small to change this integral, which, in turn, leads to the conservation of $J$ in accordance with (5.2). Thus, $J$ is the two-dimensional vorticity analogue of Corrsin's invariant.

We note, in passing, that the conservation of $J$ suggests that, following the arguments of § 4.2, the wavenumber characteristic of the intersection of the $E \sim Jk^{-1}$ and $E \sim \hat{I}k^3$ regions, $k^*$, scales as $k^* \sim (J/\hat{I})^{1/4} \sim t^{-m}$, $m \sim 0.6$. That is, if $E \sim Jk^{-1}$ at $t = 0$, then the transient growth of $\hat{I}$ will lead to $E \sim Jk^{-1} + \hat{I}k^3/4$, and if $\hat{I}$ grows as $\hat{I} \sim t^{2.5}$, as in Ossai & Lesieur (2001), then $k^* \sim t^{-0.6}$. In this case, $k^*$ decreases at roughly the same rate as the inverse of the integral scale. This contrasts with $E \sim Lk$ spectra, where $k^*$ decreases considerably faster than $\ell^{-1}$, leading to the linear part of the spectrum being progressively overshadowed by the $Ik^3$ part.

## 6. The large-scale dynamics of $E \sim IK^3$ turbulence

We complete our discussion of dynamics with a brief review of the case where $J = L = 0$, and hence $E(k \to 0) \sim Ik^3$. Our main interest here lies in estimating the rate of change of Loitsyansky's integral, $I$, which is responsible for the gradual disappearance of the $E \sim Lk$ part of the spectrum, as discussed in § 4.2.

There are no exact results relating to the rate of change of $I$, only closure estimates. Perhaps the most common closure is EDQNM, but its predictive value in two-dimensional turbulence is limited, partly because of the existence of coherent vortices. A simpler, though still unsatisfactory, approach appears in Davidson (2004), which is a direct extension of the analysis of three-dimensional turbulence by Batchelor & Proudman (1956). Here the central, and indeed only, assumption is that fourth-order cumulants are exponentially small at large separation. This is tantamount to saying that there are negligible long-range interactions in the fourth-order statistics. Note that this is a much weaker assumption than quasi-normality, which requires that fourth-order cumulants are zero for arbitrary separation. The analysis predicts (Davidson 2004, § 10.3),

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2\int \langle ss'\rangle \,\mathrm{d}\boldsymbol{r}, \quad s = u_x^2 - u_y^2, \tag{6.1}$$

or equivalently,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = Cu^4 l^2,$$

for some dimensionless pre-factor, $C$. If the large scales evolve in a self-similar manner (and it is not certain that they do), then $C$ is independent of time. Given that the integral scale is observed to grow approximately as $\ell \sim t^{0.5\pm0.05}$ (Lowe & Davidson 2005) while $u$ is approximately conserved, this suggests $I \sim t^{3.0}$, which is, perhaps, not so far from the $I \sim t^{2.5}$ growth observed in Ossai & Lesieur (2001). It is possible, however, that this is just coincidence. Certainly, the central assumption in this closure, that fourth-order cumulants can be ignored at large separation, must be checked independently.

Perhaps the most reliable way of estimating the rate of growth of $I(t)$ is to return to (2.26),

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 4\pi u^3 [r^3 K]_\infty.$$

Given that $K_\infty = a(r/\ell)^{-3} + O(r^{-4})$, for some pre-factor $a$, this yields

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 4\pi a u^3 \ell^3. \tag{6.2}$$

The observed $\sqrt{t}$ growth in $\ell$ then suggests $I \sim t^{2.5}$, which is consistent with Ossai & Lesieur (2001). Note that this strong growth in $I(t)$ lies in marked contrast to three dimensions, where $I$ is found to be more or less constant in fully developed turbulence (Ishida *et al.* 2006).

Before leaving the subject of $E \sim Ik^3$ turbulence, we ask if there is some two-dimensional analogue of (1.7), relating $I$ to the angular momentum of the flow. It turns out that there is, though the link is somewhat tenuous. To demonstrate this, we follow the original argument of Landau, adapted from three dimensions to two, and start with inhomogeneous turbulence in a large closed circular domain of radius $R$. It is readily verified that the net angular momentum in such a closed domain is

$$\boldsymbol{H} = \int_V (\boldsymbol{x} \times \boldsymbol{u})\,\mathrm{d}V = 2\int_V \psi\,\mathrm{d}V \hat{\boldsymbol{e}}_z, \tag{6.3}$$

and so, on squaring and averaging, we have,

$$\langle \boldsymbol{H}^2 \rangle = 4 \iint \langle \psi \psi' \rangle \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}', \tag{6.4}$$

which is reminiscent of (2.31):

$$I = 4 \int \langle \psi \psi' \rangle \,\mathrm{d}\boldsymbol{r}. \tag{6.5}$$

However, to make the link between $I$ and $\boldsymbol{H}$, we have the same problem as Landau had in three dimensions. We have to assume that remote points are statistically independent, so that, as $R \to \infty$ , only a small volume of fluid near the boundary is influenced by that boundary. It is then possible to show that

$$I = 4 \int \langle \psi \psi' \rangle \,\mathrm{d}\boldsymbol{r} = \operatorname*{Lim}_{V \to \infty} \frac{\langle \boldsymbol{H}^2 \rangle}{V}, \tag{6.6}$$

which is the two-dimensional counterpart of (1.7). However, as in three dimensions, the argument is unreliable because remote points are not, in general, statistically independent, and indeed we expect $\langle \psi \psi' \rangle_\infty \sim r^{-3}$.

## 7. Relating statistics to structure

We close by discussing the relationship between our findings and the structure of the evolving vorticity field seen in numerical simulations. These simulations typically have periodic boundary conditions and start with Fourier modes of random phase. Often they are initialized with an energy spectrum of the form $E(k \to 0) \sim Ik^3$. After an initial transient, the vorticity field develops a filamentary structure, similar to that of a passive scalar, as the enstrophy cascades to small scales. However, embedded within this sea of sinuous vortex filaments, we observe long-lived coherent vortices, often in the form of near-circular monopoles (see, for example, McWilliams 1984, or Benzi, Patarnello & Santangelo 1988). Thus, there are two generic types of structure, filaments and coherent vortices, and these are characterized by different types of dynamics (Bartello & Warn 1996). The filaments are continually shredded and swept around by the chaotic velocity field, feeding the enstrophy cascade in accordance with the classical theory of Batchelor (1969), while the coherent vortices retain their size, shape and strength over long periods of time, largely insulated from the strain field of the other vortices. Coherent vortices occasionally die or increase in size and strength through collisions or mergers with other coherent vortices, but this occurs over a time scale somewhat longer than that which characterizes the enstrophy cascade (McWilliams 1984). This has led many researchers to suggest that the late stages of decay may be characterized as a dilute gas of monopole vortices (Carnevale *et al.* 1991).

Perhaps the first point to note is that, provided $L_{BOX} \gg \ell$, periodicity does not exclude spectra of the form $E \sim Lk$ or $E \sim Jk^{-1}$. That is, the fact that $\int \boldsymbol{u}\, \mathrm{d}V = 0$ and $\int \omega\, \mathrm{d}V = 0$, when evaluated over the whole domain, does not exclude the possibility that $\int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle\, \mathrm{d}\boldsymbol{r}$, or $\int \langle \omega\omega' \rangle\, \mathrm{d}\boldsymbol{r}$, are finite, as discussed in § 1.1. Now the regions of the flow which are dominated by vortex filaments of mixed sign will have negligible dipole or monopole moments, and so it is probable that they make an $E \sim Ik^3$ contribution to the energy spectrum at small $k$. Thus, the observed persistence of the $E \sim Ik^3$ part of the spectrum is, perhaps, understandable. What is more difficult to interpret, however, is the observation that the long-term dynamics are dominated by a random sea of monopoles. Such a distribution appears, at first sight, to require $E(k \to 0) \sim Jk^{-1}$. However, nearly all simulations in large domains show $E(k \to 0) \sim Ik^3$ (see, for example, Chasnov 1997; Ossai & Lesieur, 2001; Lowe & Davidson 2005). Moreover, we have seen that both $I$ and $J$ are invariants in isotropic turbulence, and so they are presumably also conserved in periodic simulations, provided, of course, that the domain is large enough. So, if the initial conditions are such that $L = J = 0$, i.e. $E(k \to 0) \sim Ik^3$, we can never develop an $E \sim Jk^{-1}$ spectrum. Does this exclude an end state composed of a sea of monopoles? It turns out that it does not.

In fact, there is no contradiction between initial conditions in which $J = 0$ and the emergence of monopoles, as can be seen from the following argument. Consider a statistically homogeneous flow composed of a sea of monopoles, with $\langle \omega \rangle = 0$. If the monopoles happened to be statistically independent, free of any mechanistic constraint, then the central limit theorem would give

$$\int_V \omega\, \mathrm{d}V \sim V^{1/2}, \tag{7.1}$$

for some large volume $V$, thus ensuring a non-zero value of $J$. That is, in a large but finite volume, enclosing a collection of randomly located monopoles, there will not be perfect cancellation of the positive and negative monopoles, even though $\langle \omega \rangle = 0$.

Thus, viewed from a distance, the contents of the volume itself looks like a monopole, rather than a dipole. As discussed in §2, the corresponding velocity field has an infinite energy density.

Now consider a field of homogenous turbulence which has emerged from an initial condition in which $J$ is zero. Such a flow is subject to a powerful constraint: its kinetic energy density must remain finite, and so it cannot approach a state in which $J$ is finite and $\int \omega \, dV \sim V^{1/2}$. Thus, if monopole vortices emerge in the late stages of such turbulence, they must organize themselves in such a way so as to keep $\langle \boldsymbol{u}^2 \rangle$ finite and avoid (7.1). For example, if the initial state consists of a random sea of dipoles, which then fracture to form a sea of monopoles (Couder & Basdevant 1986) then the resulting motion of the dipoles is constrained so as to keep $\langle \boldsymbol{u}^2 \rangle$ finite. This, in turn, imposes a constraint on the relative spatial locations which the monopoles can adopt, imposing some degree of coupling between the positive and negative monopoles. In particular, the pairing of monopoles of opposite sign must be sufficiently organized as to ensure that the net vorticity in a large but finite volume, $V$, grows more slowly than $V^{1/2}$ as $V$ increases, thus avoiding (7.1). It is almost as if the need to keep $\langle \boldsymbol{u}^2 \rangle$ finite ensures that the monopoles are more thoroughly mixed than would be the case if they were randomly located. In such a situation, a naive application of the central limit theorem fails.

In short, while a $E \sim Jk^{-1}$ spectrum suggests that, within the turbulence, we have a sea of random monopoles, the converse need not be true: a sea of monopoles need not imply an $E \sim Jk^{-1}$ spectrum.

## 8. Conclusions

It is usually assumed that, for small $k$, the energy spectrum for homogeneous, two-dimensional turbulence scales as $E(k \to 0) \sim Ik^3$. Here, we have examined two possible alternatives, $E(k \to 0) \sim Lk$ and $E(k \to 0) \sim Jk^{-1}$. The $E \sim Lk$ scaling is shown to be entirely natural, the only prerequisite being that typical turbulent eddies posses a finite amount of linear impulse. As in three dimensions, $L$ is an invariant, and its invariance is a direct consequence of the principle of conservation of linear momentum. The $E \sim Jk^{-1}$ scaling, on the other hand, is somewhat pathological, as $\langle \boldsymbol{u}^2 \rangle$ diverges in the strictly isotropic case. However, the scaling $E \sim Jk^{-1}$ may be realized in numerical simulations in period squares, where the lower cutoff in $k$ keeps the energy finite. In any event, $E \sim Jk^{-1}$ is the hallmark of a sea of randomly located monopole vortices. The pre-factor $J$ is, like $L$, an invariant, similar in nature to Corrsin's invariant in passive scalar mixing.

The author would like to thank Peter Bartello, whose observation that $E \sim k^{-1}$ spectra are realizable in numerical simulations provided much of the motivation for this study. The author is also grateful to Peter Bartello for making available examples of numerical simulations where the initial conditions consisted of monopoles. The comments of three anonymous referees proved useful.

REFERENCES

Batchelor, G. K. 1969 Computation of the energy spectrum in two-dimensional turbulence. *Phys. Fluids Suppl.* II, **12,** 233.

Batchelor, G. K. & Proudman, I. 1956 The large-scale structure of homogenous turbulence. *Phil. Trans. R. Soc. Lond.* A **248**, 369.

BARTELLO, P. & WARN, D. 1996 Self-similarity of decaying two-dimensional turbulence. *J. Fluid Mech.* **326**, 357.

BENZI, R., PATARNELLO, S. & SANTANGELO, P. 1988 Self-similar coherent structures in two-dimensional decaying turbulence. *J. Phys.* A **21**, 1221.

CARNEVALE, G. F., MCWILLIAMS J. C., POMEAU, Y., WEISS, J. B. & YOUNG, W. R. 1991 Evolution of vortex statistics in two-dimensional turbulence. *Phys. Rev. Lett.* **66**, 2735.

CHASNOV, J. R. 1997 On the decay of two-dimensional homogeneous turbulence. *Phys. Fluids* **9**(1), 171.

COUDER, Y. & BASDEVANT, C. 1986 Experimental and numerical study of vortex couples in two-dimensional flows. *J. Fluid Mech.* **173**, 225.

DAVIDSON, P. A. 2004 *Turbulence: An Introduction for Scientists and Engineers*. Oxford University Press.

ISHIDA, T., DAVIDSON, P. A. & KANADA, Y. 2006 On the decay of isotropic turbulence. *J. Fluid Mech.* **564**, 455.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*, 1st edn. Pergamon.

LESIEUR, M. & HERRING, J. 1985 Diffusion of a passive scalar in two-dimensional turbulence. *J. Fluid Mech.* **161**, 77.

LESIEUR, M., OSSAI, S. & METAIS, O. 1999 Infrared pressure spectra in 3D and 2D isotropic turbulence. *Phys. Fluids* **11**, 1535.

LOWE, A. & DAVIDSON, P. A. 2005 The evolution of freely-decaying, isotropic, two-dimensional turbulence. *Eur. J. Mech.* B **24**, 314.

MCWILLIAMS, J. C. 1984 The emergence of isolated coherent vortices in turbulent flow. *J. Fluid Mech.* **146**, 21.

OSSAI, S. & LESIEUR, M. 2001 Large-scale energy and pressure dynamics in decaying 2D incompressible isotropic turbulence. *J. Turbulence* **2**, 172.

SAFFMAN, P. G. 1967 The large-scale structure of homogeneous turbulence. *J. Fluid Mech.* **27**, 581.

TOWNSEND, A. A. 1976 *The Structure of Turbulent Shear Flow*. 2nd edn. Cambridge University Press.